

# $k$ -colored kernels in semicomplete multipartite digraphs

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## Abstract

An  $m$ -colored digraph  $D$  has  $k$ -colored kernel if there exists a subset  $K$  of its vertices such that for every vertex  $v \notin K$  there exists an at most  $k$ -colored directed path from  $v$  to a vertex of  $K$  and for every  $u, v \in K$  there does not exist an at most  $k$ -colored directed path between them. In this paper we prove that an  $m$ -colored semicomplete  $r$ -partite digraph  $D$  has a  $k$ -colored kernel provided that  $r \geq 3$  and

- (i)  $k \geq 4$ ,
- (ii)  $k = 3$  and every  $\vec{C}_4$  contained in  $D$  is at most 2-colored and, either every  $\vec{C}_5$  contained in  $D$  is at most 3-colored or every  $\vec{C}_3 \uparrow \vec{C}_3$  contained in  $D$  is at most 2-colored,
- (iii)  $k = 2$  and every  $\vec{C}_3$  and  $\vec{C}_4$  contained in  $D$  is monochromatic.

If  $D$  is an  $m$ -colored semicomplete bipartite digraph and  $k = 2$  (resp.  $k = 3$ ) and every  $\vec{C}_4 \uparrow \vec{C}_4$  contained in  $D$  is at most 2-colored (resp. 3-colored), then  $D$  has a 2-colored (resp. 3-colored) kernel. Using these and previous results, we obtain conditions for the existence of  $k$ -colored kernels in  $m$ -colored semicomplete  $r$ -partite digraphs for every  $k \geq 2$  and  $r \geq 2$ .

*Keywords:*

$m$ -colored digraph,  $k$ -colored kernel, semicomplete multipartite digraph  
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## 1. Introduction

Let  $m, j$  and  $k$  positive integers. A digraph  $D$  is said to be  $m$ -colored if the arcs of  $D$  are colored with  $m$  colors. Given  $u, v \in V(D)$ , a directed path from  $u$  to  $v$  of  $D$ , denoted by  $u \rightsquigarrow v$ , is  $j$ -colored if all its arcs use exactly  $j$  colors and it is represented by  $u \rightsquigarrow_j v$ . When  $j = 1$ , the directed path is said to be *monochromatic*. A nonempty set  $S \subseteq V(D)$  is a  $k$ -colored *absorbent set* if for every vertex  $u \in V(D) - S$  there exists  $v \in S$  such that  $u \rightsquigarrow_j v$  with  $1 \leq j \leq k$ . A nonempty set  $S \subseteq V(D)$  is called a  $k$ -colored *independent set* if for every  $u, v \in S$  there does not exist  $u \rightsquigarrow_j v$  with  $1 \leq j \leq k$ . Let  $D$  be an  $m$ -colored digraph. A set  $K \subseteq V(D)$  is called a  $k$ -colored *kernel* if  $K$  is a  $k$ -colored absorbent and independent set. This definition was introduced in [10], where the first basic results were proved. We observe that a 1-colored kernel is a kernel by monochromatic directed paths, a notion that has widely studied in the literature, see for instance [4], [5], [6], [8], [9], [11], [12] and [13].

An arc  $(u, v) \in A(D)$  is *asymmetric* (resp. *symmetric*) if  $(v, u) \notin A(D)$  (resp.  $(v, u) \in A(D)$ ). We denote by  $\vec{C}_n$  the directed cycle of length  $n$ . A semicomplete  $r$ -partite digraph  $D$  with

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$r \geq 2$  is an orientation of an  $r$ -partite complete graph in which symmetric arcs are allowed. A digraph  $D$  is called *3-quasi-transitive* if whenever distinct vertices  $u_0, u_1, u_2, u_3 \in V(D)$  such that  $u_0 \rightarrow u_1 \rightarrow u_2 \rightarrow u_3$  there exists at least  $(u_0, u_3) \in A(D)$  or  $(u_3, u_0) \in A(D)$ . In particular, bipartite semicomplete digraphs are 3-quasi-transitive.

Let  $D'$  a subdigraph of an  $m$ -colored digraph  $D$ . We say that  $D'$  is *monochromatic* if every arc of  $D'$  is colored with the same color and  $D'$  is *at most  $k$ -colored* if the arcs of  $D'$  are colored with at most  $k$  colors. In this paper, we particularly use subdigraphs of semicomplete  $r$ -partite digraph which are at most 2- and 3-colored. We defined the digraphs  $\vec{C}_3 \uparrow \vec{C}_3$  (resp.  $\vec{C}_4 \uparrow\uparrow \vec{C}_4$ ) as two directed cycles  $\vec{C}_3$  (resp.  $\vec{C}_4$ ) joined by an arc (resp. by two consecutive arcs), see the next picture.

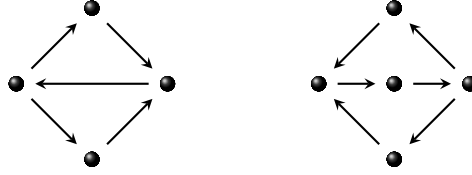


Figure 1:  $\vec{C}_3 \uparrow \vec{C}_3$  and  $\vec{C}_4 \uparrow\uparrow \vec{C}_4$ , respectively.

The goal of this work is to complete the study of the existence of  $k$ -colored kernels in semicomplete  $r$ -partite digraphs for every  $k \geq 2$ . The problem for 1-colored kernels in bipartite tournaments was studied in [8]. In that paper, the authors proved that if every  $\vec{C}_4$  contained in an  $m$ -colored bipartite tournament  $T$  is monochromatic, then  $T$  has a 1-colored kernel. Let  $r \geq 3$ . In [9], it was proved that if every  $\vec{C}_3$  and  $\vec{C}_4$  contained in a  $r$ -partite tournament  $T$  is monochromatic then  $T$  has a 1-colored kernel. In [7] among other results, we showed that  $m$ -colored quasi-transitive and 3-quasi-transitive digraphs have a  $k$ -colored kernel for every  $k \geq 3$  and  $k \geq 4$ , respectively. As a consequence,  $m$ -colored semicomplete bipartite digraphs have a  $k$ -colored kernel for every  $k \geq 3$  and  $k \geq 4$ , respectively.

In this paper we prove that an  $m$ -colored semicomplete  $r$ -partite digraph  $D$  has a  $k$ -colored kernel provided that  $r \geq 3$  and

- (i)  $k \geq 4$ ,
- (ii)  $k = 3$  and every  $\vec{C}_4$  contained in  $D$  is at most 2-colored and, either every  $\vec{C}_5$  contained in  $D$  is at most 3-colored or every  $\vec{C}_3 \uparrow \vec{C}_3$  contained in  $D$  is at most 2-colored,
- (iii)  $k = 2$  and every  $\vec{C}_3$  and  $\vec{C}_4$  contained in  $D$  is monochromatic.

If  $D$  is an  $m$ -colored semicomplete bipartite digraph and  $k = 2$  (resp.  $k = 3$ ) and every  $\vec{C}_4 \uparrow\uparrow \vec{C}_4$  contained in  $D$  is at most 2-colored (resp. 3-colored), then  $D$  has a 2-colored (resp. 3-colored) kernel. Using these and previous results, we obtain conditions for the existence of  $k$ -colored kernels in  $m$ -colored semicomplete  $r$ -partite digraphs for every  $k \geq 2$  and  $r \geq 2$  (see Corollary 4.5). If we are restricted to the family of the  $m$ -colored multipartite tournaments, then we have conditions for the existence of  $k$ -colored kernels for every  $k \geq 1$  and  $r \geq 2$  using the main results of this paper and those obtained in [8] and [9] (see Corollary 4.6).

We finish this introduction including some simple definitions and a well-known result that will be useful in proving the main results.

Let  $D$  be a digraph and  $x, y \in V(D)$ . The *distance from  $x$  to  $y$* , denoted by  $d(x, y)$  is the minimum length (number of arcs) of a  $x \rightsquigarrow y$ .

Recall that a *kernel*  $K$  of  $D$  is an independent set of vertices so that for every  $u \in V(D) \setminus K$  there exists  $(u, v) \in A(D)$ , where  $v \in K$ . We say that a digraph  $D$  is *kernel-perfect* if every nonempty induced subdigraph of  $D$  has a kernel.

Given an  $m$ -colored digraph  $D$ , we define the  $k$ -colored closure of  $D$ , denoted by  $\mathfrak{C}_k(D)$ , as the digraph such that  $V(\mathfrak{C}_k(D)) = V(D)$  and

$$A(\mathfrak{C}_k(D)) = \{(u, v) : \exists u \rightsquigarrow_j v, 1 \leq j \leq k\}.$$

**Remark 1.1.** *Observe that every  $m$ -colored digraph  $D$  has a  $k$ -colored kernel if and only if  $\mathfrak{C}_k(D)$  has a kernel.*

We will use the following theorem of P. Duchet [3].

**Theorem 1.2.** *If every directed cycle of a digraph  $D$  has a symmetric arc, then  $D$  is kernel-perfect.*

The symbol  $\triangle$  will be used to denote the end of a claim or a subclaim. We follow [2] for the general terminology on digraphs.

## 2. Preliminary results

We set  $r \geq 3$  for the rest of the paper. We denote by  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$  the partite sets of a semicomplete multipartite digraph  $D$ .

**Lemma 2.1.** *Let  $D$  be an  $m$ -colored semicomplete  $r$ -partite digraph and  $x, y \in V(D)$ . If there exists  $x \rightsquigarrow_k y$  with  $k \geq 4$  and there does not exist  $y \rightsquigarrow_{k'} x$  with  $k' \leq 4$ , then  $d(x, y) \leq 2$ .*

*Proof.* Suppose that  $x \in \mathcal{A}$  and  $y \in \mathcal{B}$ . Since there does not exist  $y \rightsquigarrow_{k'} x$ , we have that  $(x, y) \in A(D)$ . So, we assume that  $x, y \in \mathcal{A}$  and by contradiction, suppose that  $d(x, y) \geq 3$ . Consider the directed path of minimum length

$$x \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_t \longrightarrow y \quad (t \geq 2).$$

Therefore  $x_1 \in \mathcal{B}$  (with  $\mathcal{B} \neq \mathcal{A}$ ) and then  $(y, x_1) \in A(D)$ . If  $x_2 \notin \mathcal{A}$ , then  $(x_2, x) \in A(D)$  (the arc  $(x, x_2)$  implies a shorter path from  $x$  to  $y$ ). In this case, the directed path  $y \longrightarrow x_1 \longrightarrow x_2 \longrightarrow x$  is a  $y \rightsquigarrow_{k'} x$  with  $k' \leq 3$ , a contradiction (there does not exist  $y \rightsquigarrow_{k'} x$  with  $k' \leq 4$ ). Hence  $x_2 \in \mathcal{A}$  and  $t \geq 3$ , since  $x_t \notin \mathcal{A}$  and  $(x_t, y) \in A(D)$ . Recalling that  $x_2 \in \mathcal{A}$ , we get that  $x_3 \notin \mathcal{A}$  and there exists  $(x_3, x) \in A(D)$  (the arc  $(x, x_3)$  implies a shorter path from  $x$  to  $y$ ). We obtain that the directed path

$$y \longrightarrow x_1 \longrightarrow x_2 \longrightarrow x_3 \longrightarrow x$$

is a  $y \rightsquigarrow_{k'} x$  with  $k' \leq 4$ , a contradiction to the supposition of the lemma.  $\square$

**Lemma 2.2.** *Let  $D$  be an  $m$ -colored semicomplete  $r$ -partite digraph,  $k = 2$  (resp.  $k = 3$ ) and  $x, y \in V(D)$ . If there exists  $x \rightsquigarrow_k y$  and there does not exist  $y \rightsquigarrow_{k'} x$  with  $k' \leq 2$  (resp.  $k' \leq 3$ ), then  $d(x, y) \leq 4$ .*

*Proof.* Suppose that  $x \in \mathcal{A}$  and  $y \in \mathcal{B}$ . Since there does not exist  $y \rightsquigarrow_{k'} x$ , we have that  $(x, y) \in A(D)$ . So, we assume that  $x, y \in \mathcal{A}$  and by contradiction, suppose that  $d(x, y) \geq 5$ . Consider the directed path of minimum length

$$x \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_t \longrightarrow y \quad (t \geq 4).$$

If  $x_2 \notin \mathcal{A}$ , then  $(y, x_2) \in A(D)$  and  $(x_2, x) \in A(D)$  (observe that  $(x_2, y) \in A(D)$  or  $(x, x_2) \in A(D)$  implies a shorter path from  $x$  to  $y$ ). Therefore the directed path  $y \longrightarrow x_2 \longrightarrow x$  is a  $y \rightsquigarrow_{k'} x$  with  $k' \leq 2$ , a contradiction to the supposition of the lemma. Hence  $x_2 \in \mathcal{A}$  and so  $x_3 \notin \mathcal{A}$ . In a similar way as done before,  $(y, x_3) \in A(D)$  and  $(x_3, x) \in A(D)$ . It follows that the directed path  $y \longrightarrow x_3 \longrightarrow x$  is a  $y \rightsquigarrow_{k'} x$  with  $k' \leq 2$ , a contradiction to the supposition of the lemma. The proof for  $k = 3$  follows analogously.  $\square$

**Lemma 2.3.** *Let  $D$  be an  $m$ -colored semicomplete  $r$ -partite digraph such that every  $\vec{C}_4$  contained in  $D$  is at most 2-colored,  $k = 3$  and  $x, y \in V(D)$ . If there exists  $x \rightsquigarrow_k y$  and there does not exist  $y \rightsquigarrow_{k'} x$  with  $k' \leq 3$ , then  $d(x, y) \leq 2$ .*

*Proof.* Suppose that  $x \in \mathcal{A}$  and  $y \in \mathcal{B}$ . Since there does not exist  $y \rightsquigarrow_{k'} x$ , we have that  $(x, y) \in A(D)$ . So, we assume that  $x, y \in \mathcal{A}$ . By Lemma 2.2,  $d(x, y) \leq 4$ . We consider two cases.

CASE 1.  $d(x, y) = 3$ . Let  $x \rightarrow x_1 \rightarrow x_2 \rightarrow y$  be a directed path from  $x$  to  $y$ . Since  $x_1, x_2 \notin \mathcal{A}$ , we have that  $(y, x_1), (x_2, x) \in A(D)$  and so  $y \rightarrow x_1 \rightarrow x_2 \rightarrow x$  is a  $y \rightsquigarrow_{k'} x$  with  $k' \leq 3$ , a contradiction to the supposition of the lemma.

CASE 2.  $d(x, y) = 4$ . Let  $x \rightarrow x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow y$  be a directed path from  $x$  to  $y$ . If  $x_2 \notin \mathcal{A}$ , then  $(y, x_2), (x_2, x) \in A(D)$  and so  $y \rightarrow x_2 \rightarrow x$  is a  $y \rightsquigarrow_{k'} x$  with  $k' \leq 2$ , a contradiction to the supposition of the lemma. Hence  $x_2 \in \mathcal{A}$ . Notice that  $x_1, x_3 \notin \mathcal{A}$  and then  $(y, x_1), (x_3, x) \in A(D)$ . So the directed path

$$y \rightarrow x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x$$

is a  $y \rightsquigarrow_{k'} x$  with  $k' = 4$  (that is, a heterochromatic directed path from  $y$  to  $x$ ), otherwise there exists a  $y \rightsquigarrow_{k'} x$  with  $k' \leq 3$ , a contradiction to the supposition of the lemma. Therefore  $(y, x_1, x_2, x_3, y) \cong \vec{C}_4$  is at least 3-colored, a contradiction, every  $\vec{C}_4$  of  $D$  is at most 2-colored.  $\square$

**Lemma 2.4.** *Let  $D$  be an  $m$ -colored semicomplete  $r$ -partite digraph such that every  $\vec{C}_3$  and  $\vec{C}_4$  contained in  $D$  is monochromatic,  $k = 2$  and  $x, y \in V(D)$ . If there exists  $x \rightsquigarrow_k y$  and there does not exist  $y \rightsquigarrow_{k'} x$  with  $k' \leq 2$ , then  $d(x, y) \leq 2$ .*

*Proof.* Suppose that  $x \in \mathcal{A}$  and  $y \in \mathcal{B}$ . Since there does not exist  $y \rightsquigarrow_{k'} x$ , we have that  $(x, y) \in A(D)$ . So, we assume that  $x, y \in \mathcal{A}$ . By Lemma 2.2,  $d(x, y) \leq 4$ . We consider two cases.

CASE 1.  $d(x, y) = 3$ . Let  $x \rightarrow x_1 \rightarrow x_2 \rightarrow y$  be a directed path from  $x$  to  $y$ . Since  $x_1, x_2 \notin \mathcal{A}$ , we have that  $(y, x_1), (x_2, x) \in A(D)$ . Since  $(y, x_1, x_2, y), (x, x_1, x_2, x) \cong \vec{C}_3$  are monochromatic, the directed path  $y \rightarrow x_1 \rightarrow x_2 \rightarrow x$  is a monochromatic  $y \rightsquigarrow x$ , a contradiction to the supposition of the lemma.

CASE 2.  $d(x, y) = 4$ . Let  $x \rightarrow x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow y$  be a directed path from  $x$  to  $y$ . If  $x_2 \notin \mathcal{A}$ , then  $(y, x_2), (x_2, x) \in A(D)$  and so  $y \rightarrow x_2 \rightarrow x$  is a  $y \rightsquigarrow_{k'} x$  with  $k' \leq 2$ , a contradiction to the supposition of the lemma. Hence  $x_2 \in \mathcal{A}$ . Notice that  $x_1, x_3 \notin \mathcal{A}$  and then  $(y, x_1), (x_3, x) \in A(D)$ . Since  $(y, x_1, x_2, x_3, y), (x, x_1, x_2, x_3, x) \cong \vec{C}_4$  are monochromatic, the directed path  $y \rightarrow x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x$  is a monochromatic  $y \rightsquigarrow x$ , a contradiction to the supposition of the lemma.  $\square$

Analogously, we can prove the following lemma in case of semicomplete bipartite digraphs.

**Lemma 2.5.** *Let  $D$  be an  $m$ -colored semicomplete bipartite digraph such that every  $\vec{C}_4 \uparrow \vec{C}_4$  contained in  $D$  is at most  $k$ -colored,  $k = 2$  (resp.  $k = 3$ ) and  $x, y \in V(D)$ . If there exists  $x \rightsquigarrow_k y$  and there does not exist  $y \rightsquigarrow_{k'} x$  with  $k' \leq 2$  (resp.  $k' \leq 3$ ), then  $d(x, y) \leq 2$ .*

### 3. Flowers, cycles and closed walks in the $k$ -colored closure of semicomplete $r$ -partite digraphs

To begin with, we define the *flower*  $F_s$  with  $s$  petals as the digraph obtained by replacing every edge of the star  $K_{1,s}$  by a symmetric arc. If every edge of the complete graph  $K_n$  is replaced by a symmetric arc, then the resulting digraph  $D$  on  $n$  vertices is symmetric semicomplete.

**Remark 3.1.** *Let  $D$  be an  $m$ -colored digraph isomorphic to a  $\vec{C}_3$  or a flower  $F_s$  such that  $s \geq 1$ . Then  $\mathfrak{C}_k(F_s)$  with  $k \geq 2$  is a symmetric semicomplete digraph.*

This section is devoted to detail the common beginning of the proofs of Theorems 4.1 - 4.4 in the next section. The procedure is similar to that employed in the proof of Theorem 7 of [7]. We include it here to make this work self-contained. In every case, we apply Theorem 1.2 and Remark 1.1 to show that every directed cycle of the  $k$ -colored closure  $\mathfrak{C}_k(D)$  of the corresponding digraph  $D$  has a symmetric arc and we proceed by contradiction.

First, we make a sketch of the following procedure in general terms. We suppose that there exists a directed cycle  $\gamma$  in  $\mathfrak{C}_k(D)$  without symmetric arcs and using Lemmas 2.1, 2.3, 2.4 and 2.5 according to each specific case, we prove that every arc of  $\gamma$  corresponds to an arc or a directed path of length 2 in the original digraph  $D$ . At this point, we consider the closed walk  $\delta$ , subdigraph of  $D$ , constructed by the concatenation of the already mentioned arcs or directed paths of length 2 and study its properties. In the next step, we define a closed subwalk  $\varepsilon$  of  $\delta$  satisfying some prefixed properties. Then, we show that this subdigraph of  $\delta$  exists and can be described in a neat form.

Formally, let  $D$  be an  $m$ -colored semicomplete  $r$ -partite digraph with  $r \geq 2$ . By contradiction, suppose that  $\gamma = (u_0, u_1, \dots, u_p, u_0)$  is a cycle in  $\mathfrak{C}_k(D)$  without any symmetric arc. Observe that if  $p = 1$ , then  $\gamma$  has a symmetric arc and we are done. So, assume that  $p \geq 2$ . Let  $x$  and  $y$  be two consecutive vertices of  $\gamma$ . Consider the following instances recalling that  $\gamma$  has no symmetric arcs:

- (a)  $k \geq 4$ . The conditions of Lemma 2.1 are satisfied and we conclude that  $d(x, y) \leq 2$ .
- (b)  $k = 3$  and every  $\vec{C}_4$  contained in  $D$  is at most 2-colored. The conditions of Lemma 2.3 are satisfied and we conclude that  $d(x, y) \leq 2$ .
- (c)  $k = 2$  and every  $\vec{C}_3$  and  $\vec{C}_4$  contained in  $D$  is monochromatic. The conditions of Lemma 2.4 are satisfied and we conclude that  $d(x, y) \leq 2$ .
- (d)  $r = 2$ ,  $k = 2$  or 3 and every  $\vec{C}_4 \uparrow \vec{C}_4$  contained in  $D$  is at most  $k$ -colored. The conditions of Lemma 2.5 are satisfied and we conclude that  $d(x, y) \leq 2$ .

Therefore, in any case we can assume that every arc of  $\gamma$  corresponds to an arc or a directed path of length 2 in  $D$ . Let  $\delta$  be the closed directed walk defined by the concatenation of the arcs and the directed paths of length 2 corresponding to the arcs of  $\gamma$ .

**Remark 3.2.** *There exist at least two consecutive vertices of  $\gamma$  in every directed walk of length at least 3 of  $\delta$ .*

The following lemma settles two simple properties of  $\delta$ .

**Lemma 3.3.** *Let  $\delta$  be defined as before. Therefore*

- (i)  $\delta$  contains a directed path of length at least 3 and
- (ii) there are neither  $\vec{C}_3$  nor flowers  $F_s$  with  $s \geq 2$  in  $\delta$ .

*Proof.* For the first claim, suppose that every directed path of  $\delta$  has length at most 2. Then, either  $\delta$  contains a  $\vec{C}_3$  or  $\delta$  is isomorphic to a flower  $F_s$  with  $s \geq 2$  and by Remarks 3.1 and 3.2,  $\gamma$  has a symmetric arc, a contradiction. For the second, observe that if  $\delta$  contains  $\vec{C}_3$  or a flower  $F_s$  with  $s \geq 2$ , then by Remark 3.2, two consecutive vertices  $u_i$  and  $u_{i+1}$  of  $\gamma$  (the subindices are taken modulo  $p$ ) belong to the vertices of a  $\vec{C}_3$  or a flower  $F_s$ , respectively. So, by Remark 3.1, there exists a symmetric arc between  $u_i$  and  $u_{i+1}$  of  $\gamma$ , which is a contradiction.  $\square$

Let  $\delta = (y_0, y_1, \dots, y_s)$ .

**Remark 3.4.** (i) *If there exists a flower  $F_s$  in  $\delta$ , then  $s = 1$ .*

(ii) *There are no consecutive vertices of  $\gamma$  in a flower.*

(iii) *If there exists a subdigraph  $y_j \rightarrow y_{j+1} \rightarrow y_{j+2} \rightarrow y_{j+3} \rightarrow y_{j+4}$  of  $\delta$ , where  $y_{j+1} = y_{j+3}$  (that is,  $y_{j+1} \longleftrightarrow y_{j+2}$  is a flower), then  $y_{j+2} \in V(\gamma)$ .*

Notice that if  $\delta = \gamma$ , as we will see, the same argument of the proof will work even easier.

Observe that there exist  $y_{i_0}, y_{i_1}, \dots, y_{i_p} \in V(\delta)$  such that  $i_j < i_{j+1}$  and  $u_l = y_{j_l}$ , where  $0 \leq l \leq p$ .

We define  $\varepsilon = (y_i, y_{i+1}, \dots, y_{i+l})$  of minimum length ( $0 \leq i \leq s$  and the indices are taken modulo  $s+1$ ) such that

- (i)  $y_i = y_{i+l}$ ,  $l \geq 3$ ,
- (ii)  $y_i \neq y_t$  for  $i+1 \leq t \leq i+l-1$ ,
- (iii) if  $y_q = y_r$ , then  $r = q+2$  ( $i+1 \leq q, r \leq i+l-1$ ),
- (iv) there exist  $y_{i_1}, y_{i_2}, \dots, y_{i_{k+1}} \in V(\varepsilon)$  such that  $y_{i_1} = u_j$ ,  $y_{i_2} = u_{j+1}$ ,  $\dots$ ,  $y_{i_{k+1}} = u_{j+k}$  with  $k \geq 1$ , and
- (v)  $y_{i+1} \neq y_{i+l-1}$ .

**Lemma 3.5.** *There exists  $\varepsilon$  a closed subwalk of  $\delta$ .*

*Proof.* Since  $\delta$  is a closed walk,  $p \geq 2$  and using Lemma 3.3(i), condition (i) is satisfied. For (ii), if there exists  $t < l$  such that  $y_i = y_{i+t}$ , then, by the minimality of  $\varepsilon$ ,  $t = 2$  and  $l - t = 2$  and therefore  $(i+l) - (i+t) = 2$ . By (i), we have that  $y_i = y_{i+t} = y_{i+l}$  and so  $l = 4$ . We obtain that

$$y_{i+1} \longleftrightarrow y_i = y_{i+2} = y_{i+4} \longleftrightarrow y_{i+3},$$

which is a flower  $F_2$  in  $\delta$ , a contradiction to Lemma 3.3(ii). Condition (iii) follows from the minimality of  $\varepsilon$  and condition (iv) is immediate from the definition of  $\delta$  and the fact that  $l \geq 3$ . If  $y_{i+1} = y_{i+l-1}$ , then by (iii),  $l = 4$  and hence

$$y_{i+4} = y_i \longleftrightarrow y_{i+1} = y_{i+3} \longleftrightarrow y_{i+2}$$

which is a flower  $F_2$  in  $\delta$ , a contradiction to Lemma 3.3(ii). Condition (v) follows.  $\square$

Since  $\delta$  is not a flower itself by supposition, we can establish the structure of  $\varepsilon$  with precision.

**Corollary 3.6.** *The closed subwalk  $\varepsilon$  of  $\delta$  is a directed cycle of length at least 3 with perhaps symmetric arcs attached to some vertices (maybe none) of the cycle for which the exterior endpoints are vertices of  $\gamma$ .*

An example of  $\varepsilon$  is depicted in [7].

For the sake of a clearer exposition of the forthcoming proofs in the next section, let us rename  $\varepsilon = (y_0, y_1, \dots, y_l)$ . By (v) of the definition of  $\varepsilon$ , we have that  $y_1 \neq y_{l-1}$  and by (iv), there exist consecutive  $u_0, u_1, \dots, u_k \in V(\gamma)$  in  $\varepsilon$  with  $k \geq 1$ . Notice that  $u_0$  and  $u_k$  could not be consecutive vertices of  $\gamma$  and similarly,  $(y_{l-1}, y_0) \in A(\varepsilon)$  could not be an arc of  $\gamma$ . Let  $u_1 = y_i$  be the second vertex of  $\gamma$  from  $y_0$ . Observe that  $1 \leq i \leq 3$  by the definition of  $\varepsilon$  and either  $u_0 = y_0$  ( $1 \leq i \leq 2$ ) or  $u_0 = y_1$  ( $2 \leq i \leq 3$ ).

Let us suppose that there exists  $(u_1, y_0) \in A(D)$ , then

- (i) if  $u_0 = y_0$ , then  $(u_1, u_0) \in A(D)$  and  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ , a contradiction,
- (ii) if  $u_0 = y_1$ , then we have that  $u_1 \longrightarrow y_0 \longrightarrow y_1 = u_0$  and we arrive at the contradiction of (i).

Therefore, without loss of generality we can assume that

$$(u_1, y_0) \notin A(D). \quad (\nabla)$$

#### 4. Main theorems

First, we recall that by supposition  $\gamma = (u_0, u_1, \dots, u_p, u_0)$  is a cycle in  $\mathfrak{C}_k(D)$  without any symmetric arc and  $\varepsilon = (y_0, y_1, \dots, y_l)$  is a closed subwalk of  $\delta$ . The beginning of every proof of the following theorems are the arguments stated in Section 3.

**Theorem 4.1.** *Let  $D$  be an  $m$ -colored semicomplete  $r$ -partite digraph and  $k \geq 4$ . Then  $D$  has a  $k$ -colored kernel.*

*Proof.* In this case, we use instance (a) to assume that every arc of  $\gamma$  corresponds to an arc or a directed path of length 2 in  $D$ .

**Claim.**  $(u_1, y_j) \in A(D)$  for some  $l - 2 \leq j \leq l - 1$ .

*Proof of the claim.* To prove the claim, suppose by contradiction that  $q$  is the maximum index such that  $(u_1, y_q) \in A(D)$  with  $q \leq l - 3$ . Consider the directed path

$$u_1 \longrightarrow y_q \longrightarrow y_{q+1} \longrightarrow y_{q+2} \longrightarrow y_{q+3}$$

(observe that  $q + 3 \leq l$ ). First, we will show that  $u_1$  and  $y_{q+3}$  belong to the same part of  $D$ . If  $(u_1, y_{q+3}) \in A(D)$ , then  $q + 3 = l$  because  $q$  is maximum and  $y_{q+3} = y_0$ , a contradiction to  $(\nabla)$ . If  $(y_{q+3}, u_1) \in A(D)$ , then we have the directed cycle  $(u_1, y_q, y_{q+1}, y_{q+2}, y_{q+3}, u_1)$ .

If  $y_q = u_t$  for some  $2 \leq t \leq p$ , then either  $y_{q+1} = u_{t+1}$  or  $y_{q+2} = u_{t+1}$ . Therefore there exists  $u_{t+1} \rightsquigarrow_k u_t$  with  $k \leq 4$ , a contradiction,  $\gamma$  has a symmetric arc between  $u_t$  and  $u_{t+1}$ . Analogously, if  $y_{q+1} = u_t$  and either  $y_{q+2} = u_{t+1}$  or  $y_{q+3} = u_{t+1}$ , then we arrive to the same contradiction as before. Finally, if  $y_{q+1} = u_0$ , then there exists  $u_1 \rightsquigarrow_k u_0$  with  $k \leq 4$ , a contradiction,  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ . We conclude that  $u_1, y_{q+3} \in \mathcal{A}$  (the same part of the semicomplete  $r$ -partite digraph  $D$ ). As a consequence,  $y_q, y_{q+2} \notin \mathcal{A}$  and there exists  $(y_{q+2}, u_1) \in A(D)$  by the maximality of  $q$ .

We obtain the directed cycle  $\vec{C}_4 \cong (y_q, y_{q+1}, y_{q+2}, u_1, y_q)$  in which there are no two consecutive vertices of  $\gamma$ , otherwise  $\gamma$  has a symmetric arc, a contradiction. Hence,  $y_{q+1} = u_t$  and  $y_{q+3} \in V(\gamma)$ . Since there exists  $u_1 \rightsquigarrow_k y_{q+3}$  with  $k = 4$ , we have that  $y_{q+3} \neq u_0$ , otherwise  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ . Then  $y_{q+3} = u_{t+1}$  and thus  $q + 3 < l$  and we consider the extended directed path

$$u_1 \longrightarrow y_q \longrightarrow y_{q+1} = u_t \longrightarrow y_{q+2} \longrightarrow y_{q+3} = u_{t+1} \longrightarrow y_{q+4}.$$

Recall that  $u_1, y_{q+3} \in \mathcal{A}$  and then  $y_{q+4} \notin \mathcal{A}$ . By the maximality of  $q$  and since  $(u_1, y_0) \notin A(D)$ , there exists  $(y_{q+4}, u_1) \in A(D)$ . We obtain the directed path

$$u_{t+1} = y_{q+3} \longrightarrow y_{q+4} \longrightarrow u_1 \longrightarrow y_q \longrightarrow y_{q+1} = u_t,$$

a contradiction, there exists a symmetric arc between  $u_t$  and  $u_{t+1}$  in  $\gamma$ . The claim is proved.  $\triangle$

We conclude the proof of the theorem applying the Claim. In the worst case, we have that  $q = l - 2$  and  $y_1 = u_0$ . We obtain the directed cycle

$$\vec{C}_4 \cong (u_1, y_q = y_{l-2}, y_{l-1}, y_0, y_1 = u_0, u_1)$$

and there exists  $u_1 \rightsquigarrow_k u_0$  with  $k = 4$ , a contradiction,  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ . In any other case, there exists  $u_1 \rightsquigarrow_k u_0$  with  $k \leq 4$  and it yields the same contradiction as before.  $\square$

**Theorem 4.2.** *Let  $D$  be an  $m$ -colored semicomplete  $r$ -partite digraph and  $k = 3$ . If every  $\vec{C}_4$  contained in  $D$  is at most 2-colored and, either every  $\vec{C}_5$  contained in  $D$  is at most 3-colored or every  $\vec{C}_3 \uparrow \vec{C}_3$  contained in  $D$  is at most 2-colored, then  $D$  has a 3-colored kernel.*

*Proof.* In this case, we use instance (b) to assume that every arc of  $\gamma$  corresponds to an arc or a directed path of length 2 in  $D$ .

**Claim.**  $(u_1, y_j) \in A(D)$  for some  $l - 2 \leq j \leq l - 1$ .

*Proof of the claim.* To prove the claim, suppose by contradiction that  $q$  is the maximum index such that  $(u_1, y_q) \in A(D)$  with  $q \leq l - 3$ . Consider the directed path

$$u_1 \longrightarrow y_q \longrightarrow y_{q+1} \longrightarrow y_{q+2}.$$

First, we will show that  $u_1$  and  $y_{q+2}$  belong to the same part of  $D$ . Observe that  $(u_1, y_{q+2}) \in A(D)$  is impossible by the choice of  $q$  and since  $q+2 < l$ . Therefore, we suppose that  $(y_{q+2}, u_1) \in A(D)$ . If there exist  $u_t, u_{t+1} \in V(\gamma)$  (indices are taken modulo  $p+1$ ), such that  $u_t, u_{t+1} \in \{y_q, y_{q+1}, y_{q+2}\}$ , then there exists  $u_{t+1} \rightsquigarrow_k u_t$  with  $k \leq 3$ , a contradiction,  $\gamma$  has a symmetric arc between  $u_t$  and  $u_{t+1}$ . Hence  $y_{q+1} = u_t$  and  $y_q, y_{q+2} \notin V(\gamma)$ . Since the directed cycle  $\vec{C}_4 \cong (u_1, y_q, y_{q+1}, y_{q+2}, u_1)$  is at most 2-colored, the directed path

$$u_1 \longrightarrow y_q \longrightarrow y_{q+1} \longrightarrow y_{q+2} \longrightarrow y_{q+3}$$

is at most 3-colored and then  $y_{q+3} \neq u_0$  and  $y_{q+3} = u_{t+1}$  (in virtue of the definition of  $\varepsilon$ ). Moreover,  $q+3 < l$ . Let us suppose that there exists an arc between  $y_{q+3}$  and  $u_1$ . By the maximality of  $q$ , we have that  $(y_{q+3}, u_1) \in A(D)$ . Then

$$y_{q+3} = u_{t+1} \longrightarrow u_1 \longrightarrow y_q \longrightarrow y_{q+1} = u_t$$

is an at most 3-colored directed path, a contradiction,  $\gamma$  has a symmetric arc between  $u_t$  and  $u_{t+1}$ . In consequence,  $u_1, y_{q+3} \in \mathcal{A}$  (a same part of  $D$ ),  $y_q \notin \mathcal{A}$  and there exists an arc between  $y_q$  and  $y_{q+3}$ .

If  $(y_{q+3}, y_q) \in A(D)$ , then the directed path

$$y_{q+3} = u_{t+1} \longrightarrow y_q \longrightarrow y_{q+1} = u_t$$

is an at most 3-colored directed path, a contradiction,  $\gamma$  has a symmetric arc between  $u_t$  and  $u_{t+1}$ . Thus,  $(y_q, y_{q+3}) \in A(D)$  and let us consider the extended directed path

$$u_1 \longrightarrow y_q \longrightarrow y_{q+1} = u_t \longrightarrow y_{q+2} \longrightarrow y_{q+3} = u_{t+1} \longrightarrow y_{q+4},$$

where  $q+4 \leq l$  and furthermore,  $y_{q+4} \notin \mathcal{A}$ . There exists the arc between  $u_1$  and  $y_{q+4}$ . If  $(u_1, y_{q+4}) \in A(D)$ , then by the maximality of  $q$ ,  $y_{q+4} = y_l = y_0$  and we obtain a contradiction to the assumption  $(\nabla)$ . So,  $(y_{q+4}, u_1) \in A(D)$ .

Recalling that  $(y_{q+3}, u_1), (y_q, y_{q+3}), (y_{q+4}, u_1) \in A(D)$ , we have the directed cycles

$$\begin{aligned} \vec{C}_4 &\cong (u_1, y_q, y_{q+1} = u_t, y_{q+2}, u_1) \text{ and} \\ \vec{C}_4 &\cong (y_{q+3} = u_{t+1}, y_{q+4}, u_1, y_q, y_{q+3}) \end{aligned} \tag{1}$$

which are at most 2-colored by the condition of the theorem. Then the directed path

$$u_{t+1} = y_{q+3} \longrightarrow y_{q+4} \longrightarrow u_1 \longrightarrow y_q \longrightarrow y_{q+1} = u_t$$

is at most 3-colored given that the directed cycles of (1) have the common arc  $(u_1, y_q)$ . We have a contradiction,  $\gamma$  has a symmetric arc between  $u_t$  and  $u_{t+1}$ . We conclude that  $u_1, y_{q+2} \in \mathcal{A}$  (a same part of  $D$ ).

As a consequence,  $y_q, y_{q+1}, y_{q+3} \notin \mathcal{A}$ . The maximality of  $q$  implies that  $(y_{q+1}, u_1) \in A(D)$ . If there exists  $(u_1, y_{q+3}) \in A(D)$ , then by the maximality of  $q$ ,  $y_{q+3} = y_l = y_0$ , a contradiction to  $(\nabla)$ . So, there exists  $(y_{q+3}, u_1) \in A(D)$ .



If  $(y_q, y_{q+2}) \in A(D)$ , then there exists the directed cycle

$$\vec{C}_4 \cong (u_1, y_q, y_{q+2}, y_{q+3}, u_1)$$

which is at most 2-colored and therefore there exists  $u_{t+1} \rightsquigarrow_k u_t$  where  $k \leq 3$  and with  $u_t, u_{t+1} \in \{y_q, y_{q+1}, y_{q+2}, y_{q+3}\}$ , a contradiction,  $\gamma$  has a symmetric arc between  $u_t$  and  $u_{t+1}$ . Hence  $(y_{q+2}, y_q) \in A(D)$ .  $\triangle$

In brief, we have that  $(y_{q+1}, u_1), (y_{q+3}, u_1), (y_{q+2}, y_q) \in A(D)$  in the directed path

$$u_1 \longrightarrow y_q \longrightarrow y_{q+1} \longrightarrow y_{q+2} \longrightarrow y_{q+3}.$$

Observe that  $u_t \in \{y_q, y_{q+1}\}$ . If  $y_{q+2} = u_{t+1}$ , then  $u_{t+1} = y_{q+2} \longrightarrow y_q \longrightarrow y_{q+1}$  is at most 3-colored, a contradiction,  $\gamma$  has a symmetric arc between  $u_t$  and  $u_{t+1}$ . We conclude that  $y_{q+1} = u_t$  and either  $y_{q+3} = u_{t+1}$  or  $y_{q+3} = u_0$ . If  $y_{q+3} = u_{t+1}$ , then

$$u_{t+1} = y_{q+3} \longrightarrow u_1 \longrightarrow y_q \longrightarrow y_{q+1}$$

is at most 3-colored, a contradiction,  $\gamma$  has a symmetric arc between  $u_t$  and  $u_{t+1}$ . Thus  $y_{q+3} = u_0$ .

By condition of the theorem, every  $\vec{C}_5$  contained in  $D$  is at most 3-colored or every  $\vec{C}_3 \uparrow \vec{C}_3$  contained in  $D$  is at most 2-colored. If every  $\vec{C}_5$  is at most 3-colored, then

$$\vec{C}_5 \cong (u_1, y_q, y_{q+1}, y_{q+2}, y_{q+3} = u_0, u_1)$$

is at most 3-colored and consequently, there exists  $u_1 \rightsquigarrow_k u_0$  at most 3-colored, a contradiction,  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ .

If every  $\vec{C}_3 \uparrow \vec{C}_3$  is at most 2-colored then the  $\vec{C}_3 \uparrow \vec{C}_3$  induced by  $\{u_1, y_q, y_{q+1}, y_{q+2}\}$  is at most 2-colored and there exists  $u_1 \rightsquigarrow_k u_0$  at most 3-colored, a contradiction,  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ .

The claim is proved.  $\triangle$

To finish the proof of the theorem, we apply the Claim and consider two cases:

CASE 1.  $(u_1, y_{l-1}) \in A(D)$ . In this case the directed path  $u_1 \longrightarrow y_{l-1} \longrightarrow y_0 \longrightarrow y_1$  is at most 3-colored and we know that  $u_0 \in \{y_0, y_1\}$ . We arrive to a similar contradiction as before,  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ .

CASE 2.  $(u_1, y_{l-2}) \in A(D)$ . Observe that  $y_0 \neq u_0$  and  $y_1 = u_0$ , otherwise

$$u_1 \longrightarrow y_{l-2} \longrightarrow y_{l-1} \longrightarrow y_0 = u_0$$

is at most 3-colored, and we have the contradiction of Case 1 once more. So, we have the directed path

$$u_1 \longrightarrow y_{l-2} \longrightarrow y_{l-1} \longrightarrow y_0 \longrightarrow y_1 = u_0.$$

By assumption  $(\nabla)$ , the arc  $(y_0, u_1)$  could belong to  $A(D)$ . If that is the case, then the directed cycle

$$\vec{C}_4 \cong (u_1, y_{l-2}, y_{l-1}, y_0, u_1)$$

is at most 2-colored by the condition of the theorem and therefore,  $u_1 \rightsquigarrow_k u_0$  at most 3-colored, a contradiction,  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ . So, we suppose that  $(y_0, u_1) \notin A(D)$ . By  $(\nabla)$ , we have that  $(u_1, y_0) \notin A(D)$  and then  $u_1, y_0 \in \mathcal{A}$  and  $u_0 = y_1 \notin \mathcal{A}$ . Consequently, there exists an arc between  $y_1 = u_0$  and  $u_1$ . It is clear that  $(u_0, u_1) \in A(D)$  (otherwise we have a contradiction).

If every  $\vec{C}_5$  is at most 3-colored, then

$$\vec{C}_5 \cong (u_1, y_{l-2}, y_{l-1}, y_0, y_1 = u_0, u_1)$$

is at most 3-colored and we arrive to a similar contradiction as shown before. Hence, we can suppose that there exists a  $\vec{C}_5$  at least 4-colored and thus we assume the condition that every  $\vec{C}_3 \uparrow \vec{C}_3$  is at most 2-colored in  $D$ . Notice that  $y_{l-1} \in \mathcal{B} \neq \mathcal{A}$  and then there exists an arc between  $y_{l-1}$  and  $u_1$ . Since  $l-2$  is the maximum index such that  $(u_1, y_{l-2}) \in A(D)$ , we have that  $(y_{l-1}, u_1) \in A(D)$ . Also  $y_{l-2} \in \mathcal{C} \notin \{\mathcal{A}, \mathcal{B}\}$  and then there exists an arc between  $y_0$  and  $y_{l-2}$ . If  $(y_0, y_{l-2}) \in A(D)$ , then the  $\vec{C}_3 \uparrow \vec{C}_3$  induced by  $\{u_1, y_{l-2}, y_{l-1}, y_0\}$  is at most 2-colored. Hence there exists  $u_1 \rightsquigarrow_k u_0$  at most 3-colored, a contradiction,  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ . If  $(y_{l-2}, y_0) \in A(D)$ , then

$$u_1 \longrightarrow y_{l-2} \longrightarrow y_0 \longrightarrow y_1 = u_0$$

is at most 3-colored, a contradiction,  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ .

The theorem is proved.  $\square$

**Theorem 4.3.** *Let  $D$  be an  $m$ -colored semicomplete  $r$ -partite digraph and  $k = 2$ . If every  $\vec{C}_3$  and  $\vec{C}_4$  contained in  $D$  is monochromatic, then  $D$  has a 2-colored kernel.*

*Proof.* In this case, we use instance (c) to assume that every arc of  $\gamma$  corresponds to an arc or a directed path of length 2 in  $D$ .

**Claim 1.**  $(u_1, y_j) \in A(D)$  for some  $l-2 \leq j \leq l-1$ .

*Proof of the Claim 1.* To prove the claim, suppose by contradiction that  $q$  is the maximum index such that  $(u_1, y_q) \in A(D)$  with  $q \leq l-3$ . Without loss of generality, suppose that  $u_1 \in \mathcal{A}$ . We will need the following three subclaims.

**Subclaim 1.**  $y_{q+1} \in V(\gamma)$ .

*Proof of the Subclaim 1.* By contradiction, suppose that  $y_{q+1} \notin V(\gamma)$  and consider the directed path

$$u_1 \longrightarrow y_q \longrightarrow y_{q+1} \longrightarrow y_{q+2}$$

( $q+3 \leq l$ ). Then  $y_q, y_{q+2} \in V(\gamma)$  and without loss of generality, we can suppose that  $y_q = u_t$  and  $y_{q+1} = u_{t+1}$  for some  $2 \leq t \leq p-1$  and  $u_{t+1} \neq u_0$  (otherwise there exists  $u_1 \rightsquigarrow_k u_0$  with  $k \leq 2$ , a contradiction,  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ ). Observe that if  $y_{q+2} = u_{t+1} \notin \mathcal{A}$ , then there exists an arc between  $u_1$  and  $y_{q+2}$ . By the maximality of  $q$ , we have that  $(y_{q+2}, u_1) \in A(D)$  and then  $u_t$  and  $u_{t+1}$  are contained in the monochromatic cycle  $\vec{C}_4 \cong (u_1, y_q, y_{q+1}, y_{q+2}, u_1)$  (by hypothesis). So, there exists a monochromatic  $u_{t+1} \rightsquigarrow u_t$ , a contradiction,  $\gamma$  has a symmetric arc between  $u_t$  and  $u_{t+1}$ . Hence,  $y_{q+2} = u_{t+1} \in \mathcal{A}$ . We know that  $y_q \notin \mathcal{A}$ . Let us suppose that  $y_q \in \mathcal{B} \neq \mathcal{A}$ . There exists an arc between  $y_q$  and  $y_{q+2}$ . If  $(y_{q+2}, y_q) = (u_{t+1}, u_t) \in A(D)$ , then we arrive to the same contradiction as before. Therefore  $(y_q, y_{q+2}) = (u_t, u_{t+1}) \in A(D)$ . As a consequence  $y_{q+1}$  does not exist in  $\varepsilon$  by the definition of  $\gamma$ , a contradiction.  $\triangle$

**Subclaim 2.**  $y_{q+2} \notin V(\gamma)$ .

*Proof of the Subclaim 2.* By contradiction, suppose that  $y_{q+2} \in V(\gamma)$  and hence  $y_{q+2} = u_{t+1}$  because  $q+2 \leq l-1$ . So we have the directed path

$$u_1 \longrightarrow y_q \longrightarrow y_{q+1} = u_t \longrightarrow y_{q+2} = u_{t+1}.$$

If  $y_{q+2} = u_{t+1} \notin \mathcal{A}$ , there exists the arc  $(y_{q+2} = u_{t+1}, u_1) \in A(D)$  by the maximality of  $q$ . But then  $u_t$  and  $u_{t+1}$  are contained in a monochromatic cycle

$$\vec{C}_4 \cong (y_{q+2} = u_{t+1}, u_1, y_q, y_{q+1} = u_t, y_{q+2} = u_{t+1}),$$

a contradiction, there exists a monochromatic  $u_{t+1} \rightsquigarrow u_t$  and a symmetric arc between  $u_t$  and  $u_{t+1}$  in  $\gamma$ . Therefore,  $y_{q+2} = u_{t+1} \in \mathcal{A}$  and there exists an arc between  $y_q$  and  $y_{q+2} = u_{t+1}$  (recall that  $y_q \notin \mathcal{A}$ ). If  $(y_{q+2} = u_{t+1}, y_q) \in A(D)$ , then  $(y_{q+1} = u_t, y_{q+2} = u_{t+1}, y_q, y_{q+1} = u_t)$

is a monochromatic  $\vec{C}_3$  by hypothesis and there exists a monochromatic  $u_{t+1} \rightsquigarrow u_t$  and we get the same contradiction. Hence  $(y_q, y_{q+2} = u_{t+1}) \in A(D)$ . Consider the extended directed path

$$u_1 \longrightarrow y_q \longrightarrow y_{q+1} = u_t \longrightarrow y_{q+2} = u_{t+1} \longrightarrow y_{q+3},$$

where  $y_{q+3} \notin \mathcal{A}$  (since  $y_{q+2} = u_{t+1} \in \mathcal{A}$ ). Then, there exists an arc between  $u_1$  and  $y_{q+3}$ . If  $(u_1, y_{q+3}) \in A(D)$ , then  $y_{q+3} = y_l = y_0$ , a contradiction to  $(\nabla)$ . Thus,  $(y_{q+3}, u_1) \in A(D)$ . Recall that  $(y_q, y_{q+2} = u_{t+1}) \in A(D)$ . Hence,

$$\vec{C}_4 \cong (y_{q+2} = u_{t+1}, y_{q+3}, u_1, y_q, y_{q+2} = u_{t+1})$$

is monochromatic by hypothesis. So, there exists  $u_{t+1} \rightsquigarrow_k u_t$  with  $k \leq 2$ , a contradiction,  $\gamma$  has a symmetric arc between  $u_t$  and  $u_{t+1}$ .  $\triangle$

As a consequence of Subclaim 2, we have that  $y_{q+3} \in V(\gamma)$ .

**Subclaim 3.**  $y_{q+3} \in \mathcal{A}$ .

*Proof of the Subclaim 3.* By contradiction, suppose that  $y_{q+3} \notin \mathcal{A}$ . By Subclaim 1, we can suppose that  $y_{q+1} = u_t \in V(\gamma)$ . Consider the directed path

$$u_1 \longrightarrow y_q \longrightarrow y_{q+1} = u_t \longrightarrow y_{q+2} \longrightarrow y_{q+3}$$

Then there exists an arc between  $u_1$  and  $y_{q+3}$  (recall that  $u_1 \in \mathcal{A}$ ). By the maximality of  $q$ , we have that  $(y_{q+3}, u_1) \in A(D)$ . By Subclaim 2,  $y_{q+2} \notin V(\gamma)$  and thus  $y_{q+3} \in V(\gamma)$ . We consider two cases:

CASE 1.  $y_{q+2} \notin \mathcal{A}$ . By the maximality of  $q$ , there exists  $(y_{q+2}, u_1) \in A(D)$  and the directed cycle

$$\vec{C}_4 \cong (y_{q+2}, u_1, y_q, y_{q+1} = u_t, y_{q+2})$$

is monochromatic by hypothesis. If  $y_{q+3} = u_0$ , then there exists  $u_1 \rightsquigarrow_k u_0$  with  $k \leq 2$ , a contradiction,  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ . If  $y_{q+3} = u_{t+1}$ , then there exists  $u_{t+1} \rightsquigarrow_k u_t$  with  $k \leq 2$ , a contradiction,  $\gamma$  has a symmetric arc between  $u_t$  and  $u_{t+1}$ .

CASE 2.  $y_{q+2} \in \mathcal{A}$ . Then  $y_{q+1} \notin \mathcal{A}$  and since  $y_q \in \mathcal{B}$  (see the proof of Subclaim 1), we have that  $y_{q+1} \notin \mathcal{B}$ . Without loss of generality, suppose that  $y_{q+1} \in \mathcal{C}$ . By the maximality of  $q$ , there exists  $(y_{q+1}, u_1) \in A(D)$  and therefore the directed cycle

$$\vec{C}_3 \cong (u_1, y_q, y_{q+1} = u_t, u_1) \tag{2}$$

is monochromatic. On the other hand, there exists an arc between  $y_q$  and  $y_{q+2}$ . If  $(y_{q+2}, y_q) \in A(D)$ , then the directed cycle  $\vec{C}_3 \cong (y_{q+2}, y_q, y_{q+1} = u_t, y_{q+2})$  is monochromatic and has the same color of the  $\vec{C}_3$  of (2) because they share the arc  $(y_q, y_{q+1} = u_t) \in A(D)$ . If  $y_{q+3} = u_0$ , then there exists  $u_1 \rightsquigarrow_k u_0$  with  $k \leq 2$ , a contradiction,  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ . If  $y_{q+3} = u_{t+1}$ , then there exists  $u_{t+1} \rightsquigarrow_k u_t$  with  $k \leq 2$ , a contradiction,  $\gamma$  has a symmetric arc between  $u_t$  and  $u_{t+1}$ . So, we conclude that there exists  $(y_q, y_{q+2}) \in A(D)$ . In this case, the directed cycle

$$\vec{C}_4 \cong (u_1, y_q, y_{q+2}, y_{q+3}, u_1)$$

is monochromatic and of the same color as the  $\vec{C}_3$  of (2) because they share the arc  $(u_1, y_q) \in A(D)$ . Analogously, if  $y_{q+3} = u_0$  or  $y_{q+3} = u_{t+1}$ , we arrive to the same contradiction,  $\gamma$  has a symmetric arc.

The subclaim follows.  $\triangle$

Continuing with the proof of Claim 1, we have the following directed path

$$u_1 \longrightarrow y_q \longrightarrow y_{q+1} = u_t \longrightarrow y_{q+2} \longrightarrow y_{q+3},$$

where  $u_1, y_{q+3} \in \mathcal{A}$ ,  $y_q \in \mathcal{B}$ ,  $y_{q+2} \notin \mathcal{A}$ ,  $y_{q+2} \notin V(\gamma)$  and  $y_{q+3} \in V(\gamma)$  using Subclaims 1-3. By the maximality of  $q$ , there exists  $(y_{q+2}, u_1) \in A(D)$  creating the monochromatic directed cycle

$$\vec{C}_4 \cong (y_{q+2}, u_1, y_q, y_{q+1} = u_t, y_{q+2}). \quad (3)$$

Hence, there exists  $u_1 \rightsquigarrow_k y_{q+3}$  with  $k \leq 2$  and therefore,  $y_{q+3} \neq u_0$  (otherwise,  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ ) and then  $y_{q+3} = u_{t+1}$  with  $q+3 < l$ . Consider the extended directed path

$$u_1 \longrightarrow y_q \longrightarrow y_{q+1} = u_t \longrightarrow y_{q+2} \longrightarrow y_{q+3} = u_{t+1} \longrightarrow y_{q+4},$$

where  $y_{q+4} \notin \mathcal{A}$  (since  $y_{q+3} \in \mathcal{A}$ ). Therefore there exists an arc between  $u_1$  and  $y_{q+4}$ . If  $(u_1, y_{q+4}) \in A(D)$ , then by the maximality of  $q$ , we have that  $y_{q+4} = y_0$ , a contradiction to  $(\nabla)$ . So there exists  $(y_{q+4}, u_1) \in A(D)$ .

On the other hand, since  $y_{q+3} = u_{t+1} \in \mathcal{A}$  and  $y_q \in \mathcal{B}$ , there exists an arc between  $y_q$  and  $y_{q+3}$ . If  $(y_{q+3} = u_{t+1}, y_q) \in A(D)$ , then  $u_t$  and  $u_{t+1}$  belong to the monochromatic

$$\vec{C}_4 \cong (y_q, y_{q+1} = u_t, y_{q+2}, y_{q+3} = u_{t+1}, y_q)$$

and thus there exists a monochromatic  $u_{t+1} \rightsquigarrow u_t$ , a contradiction,  $\gamma$  has a symmetric arc between  $u_t$  and  $u_{t+1}$ . Hence  $(y_q, y_{q+3} = u_{t+1}) \in A(D)$  and we obtain the monochromatic directed cycle

$$\vec{C}_4 \cong (y_q, y_{q+3} = u_{t+1}, y_{q+4}, u_1, y_q)$$

of the same color of the  $\vec{C}_4$  of (3) because they share the arc  $(u_1, y_q) \in A(D)$ . Thus, there exists the monochromatic  $u_{t+1} = y_{q+3} \rightsquigarrow y_{q+1} = u_t$ , a contradiction,  $\gamma$  has a symmetric arc between  $u_t$  and  $u_{t+1}$ .

Claim 1 is proved.  $\triangle$

**Claim 2.**  $y_0 \in \mathcal{A}$ .

*Proof of Claim 2.* By contradiction, let us suppose that  $y_0 \notin \mathcal{A}$ . Then there exists an arc between  $u_1$  and  $y_0$ . By  $(\nabla)$ , there exists  $(y_0, u_1) \in A(D)$ . So the directed cycle

$$\vec{C}_4 \cong (y_0, u_1, y_{l-2}, y_{l-1}, y_0) \text{ or } \vec{C}_3 \cong (y_0, u_1, y_{l-1}, y_0)$$

is monochromatic and hence there exists  $u_1 \rightsquigarrow_k u_0$  with  $k \leq 2$  (recall that  $y_0 = u_0$  or  $y_1 = u_0$ ). We arrive to a contradiction,  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ , completing the proof of the claim.  $\triangle$

To finish the proof of the theorem, we consider two cases according to Claim 1.

CASE 1.  $j = l - 2$ . By Claim 2,  $y_0 \in \mathcal{A}$  and then  $y_{l-1} \notin \mathcal{A}$ . So, there exists an arc between  $y_{l-1}$  and  $u_1$ . By the maximality of  $j$ , there exists  $(y_{l-1}, u_1) \in A(D)$ . Hence the directed cycle

$$\vec{C}_3 \cong (u_1, y_{l-2}, y_{l-1}, u_1) \quad (4)$$

is monochromatic by hypothesis and then there exists  $u_1 \rightsquigarrow_k y_0$  with  $k \leq 2$ . If  $y_0 = u_0$ , then we have a contradiction,  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ . Therefore  $y_1 = u_0$ . On the other hand, there exists an arc between  $y_{l-2}$  and  $y_0$  since  $y_{l-2} \notin \mathcal{A}$  and  $y_0 \in \mathcal{A}$ . If  $(y_0, y_{l-2}) \in A(D)$ , then the directed cycle  $\vec{C}_3 \cong (y_{l-1}, y_0, y_{l-2}, y_{l-1})$  is monochromatic and of the same color as the  $\vec{C}_3$  of (4). Thus, there exists  $u_1 \rightsquigarrow_k u_0$  with  $k \leq 2$ , particularly,

$$u_1 \longrightarrow y_{l-2} \longrightarrow y_{l-1} \longrightarrow y_0 \longrightarrow y_1 = u_0,$$

we have the same contradiction as previously. In consequence,  $(y_{l-2}, y_0) \in A(D)$ . In addition, since  $y_0 \in \mathcal{A}$  and  $y_1 = u_0 \notin \mathcal{A}$ , there exists the arc  $(u_0, u_1) \in A(D)$  (otherwise,  $(u_1, u_0) \in A(D)$  yielding the same contradiction as before). Then the directed cycle

$$\vec{C}_4 \cong (y_1 = u_0, u_1, y_{l-2}, y_0, y_1 = u_0)$$

is monochromatic and there exists a monochromatic  $u_1 \rightsquigarrow u_0$ , the same contradiction once more.

CASE 2.  $j = l - 1$ . In this case, there exists the directed path  $u_1 \rightarrow y_{l-1} \rightarrow y_0$ . So,  $y_0 \neq u_0$ , otherwise we have a contradiction,  $\gamma$  has a symmetric arc between  $u_0$  and  $u_1$ . Hence  $y_1 = u_0$  and since  $y_0 \in \mathcal{A}$  by Claim 2,  $y_1 = u_0 \notin \mathcal{A}$ . Thus, there exists an arc between  $u_0$  and  $u_1$  which should be  $(u_0, u_1) \in A(D)$  (if not,  $(u_1, u_0) \in A(D)$  and we have a contradiction). Therefore the directed cycle

$$\vec{C}_4 \cong (y_1 = u_0, u_1, y_{l-1}, y_0, y_1 = u_0)$$

is monochromatic and there exists a monochromatic  $u_1 \rightsquigarrow u_0$ , the same contradiction once more.

This concludes the proof of the theorem.  $\square$

In a very similar way as the proofs of the above theorems, we can show the following theorem for semicomplete bipartite digraphs.

**Theorem 4.4.** *Let  $D$  be an  $m$ -colored semicomplete bipartite digraph and  $k = 2$  (resp.  $k = 3$ ). If every  $\vec{C}_4 \uparrow \vec{C}_4$  contained in  $D$  is at most 2-colored (resp. 3-colored), then  $D$  has a 2-colored (resp. 3-colored) kernel.*

We summarize the known results on the existence of  $k$ -colored kernels for  $m$ -colored semicomplete multipartite digraphs and multipartite tournaments in the next two corollaries.

**Corollary 4.5.** *Let  $D$  be an  $m$ -colored semicomplete  $r$ -partite digraph and  $r \geq 2$ .*

- (i) *If  $r \geq 3$ ,  $k = 2$  and every  $\vec{C}_3$  and  $\vec{C}_4$  contained in  $D$  is monochromatic, then  $D$  has a 2-colored kernel (Theorem 4.3).*
- (ii) *If  $r \geq 3$ ,  $k = 3$  and every  $\vec{C}_4$  contained in  $D$  is at most 2-colored and, either every  $\vec{C}_5$  contained in  $D$  is at most 3-colored or every  $\vec{C}_3 \uparrow \vec{C}_3$  contained in  $D$  is at most 2-colored, then  $D$  has a 3-colored kernel (Theorem 4.2).*
- (iii) *If  $r \geq 2$  and  $k \geq 4$ , then  $D$  has a  $k$ -colored kernel (Theorem 4.1 and Theorem 14 of [7]).*
- (iv) *If  $r = 2$ ,  $k = 2$  (resp.  $k = 3$ ) and every  $\vec{C}_4 \uparrow \vec{C}_4$  contained in  $D$  is at most 2-colored, then  $D$  has a 2-colored (resp. 3-colored) kernel (Theorem 4.4).*

**Corollary 4.6.** *Let  $D$  be an  $m$ -colored  $r$ -partite tournament with  $r \geq 2$ . Then the conclusions (i)–(iv) of Corollary 4.5 remain valid. Moreover,*

- (i) *if  $r = 2$ ,  $k = 1$  and every  $\vec{C}_4$  contained in  $D$  is monochromatic, then  $D$  has a 1-colored kernel (Theorem 2.1 of [8]), and*
- (ii) *if  $r \geq 3$ ,  $k = 1$  and every  $\vec{C}_3$  and  $\vec{C}_4$  contained in  $D$  is monochromatic, then  $D$  has a 1-colored kernel (Theorem 3.3 of [9]).*

We conclude this paper with the following challenging conjecture. If it were true, the resulting theorem would be a fine generalization of Theorem 3.3 proved in [9].

**Conjecture 4.7.** *Let  $D$  be an  $m$ -colored semicomplete  $r$ -partite digraph with  $r \geq 2$ . If every  $\vec{C}_3$  and  $\vec{C}_4$  contained in  $D$  is monochromatic, then  $D$  has a 1-colored kernel.*

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